A Game with Distorted Information

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This article considers a two-person game in which the first player has access to certain information that is valuable but unknown to the second player. The first player can distort the information before it is passed on to the second player. The purpose in distorting the information is to render it as useless as possible to the second player. Based on the distorted information received, the second player then maximizes some given objective. In certain cases he may still be able to use the distorted information, but sometimes the information has been so badly distorted that it becomes completely useless to him.

1. INTRODUCTION

Let Γ be a two-person zero-sum game whose parameters are described by a quadruple $\langle S, p, C, \{D_{ij}: (i, j) \in C\}\rangle$. Here S is a finite set whose elements are ordered pairs (i, j) of positive integers, p is a discrete probability function defined on S by $p_{ij} = \text{prob}\{(i, j)\}$, $C = \{(i, j) \in S: p_{ij} > 0\}$, and D_{ij} is a nonempty subset of S. This quadruple is common knowledge to the two players P_1 and P_2. The game Γ consists of three sequential moves. In the first move, a chance device uses p to pick a pair (i, j) from S. The numbers i and j are then dropped into box 1 and box 2, respectively. In the second move, P_1 peeks into the boxes to see the numbers i and j. He then closes the boxes, chooses a pair (k, l) from D_{ij}, labels k on the lid of box 1, and labels l on the lid of box 2. In the third and final move, P_2 looks at the labels on the lids, selects a box, and receives a payoff equal to the selected box's actual contents. The objective in Γ is for P_1 and P_2 to minimize and maximize, respectively, this payoff.

Notice the box contents (i, j) are true information, whereas the lid labels (k, l) are distorted information. The true information, which is useful to P_2 to select a correct box, is not accessible to him. Instead, he is given the information which has been distorted by P_1 to foil his attempt to select a correct box. The amount of distortion P_1 can introduce is specified by the sets D_{ij}. The natural questions to ask here are: How should P_1 distort the information? Is the distorted infor-

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mation still useful to P_2, and if the answer is yes, how can he use such distorted information to select a box?

The motivation of Γ comes from deception games, first considered by Thompson in his unpublished undergraduate thesis at Harvard University in 1970. Spencer [4] introduced deception games to the open literature when he posed the question of whether the value of a certain deception game is $\frac{1}{2}$. Baston and Bostock [1] settled the existence question by proving very general deception games always have a solution. The formal definition of a deception game can be found in their article [1]. In another related article, Baston and Bostock [2] formulated and solved a cover-up game. The discrete version of this cover-up game may be regarded as a particular case of the game here by taking $S = \{(i, j): i, j = 1, \ldots, n + 1\}$, $C = \{(i, j): i, j = 1, \ldots, n\}$, $p_{ij} = 1/n^2$, and $D_{ij} = \{(i, n + 1), (n + 1, j)\}$ for each (i, j) in C.

2. THE GAME

Because Γ is a finite game, its value v exists. As it is defined, Γ is a game in extensive form. The standard method to solve Γ is to convert it to its normal form, that is, to set up the appropriate game matrix. The resulting matrix game can then be solved using linear programming. The main drawback of this method is due to the exponential size of pure strategies involved. To see this is the case, let $|X|$ denote the cardinality of a set X. It is easily seen that P_1 has $\Pi\, |D_{ij}|$ pure strategies, where the product is taken over $(i, j) \in C$. Similarly, P_2 has $2^{|L|}$ pure strategies, where $L = \{(k, l): \exists (i, j) \in C \text{ such that } (k, l) \in D_{ij}\}$. For a physical interpretation, C is the set of box contents, whereas L is the set of lid labels. Because of the technical difficulty of solving very large-scale linear programs, we look for a more efficient method to solve Γ. This leads us to consider behavioral strategies that turn out to offer a computational advantage. Behavioral strategies were first introduced by Kuhn [3] in a seminal article on games in extensive form. A behavioral strategy can be viewed as a local randomization on the occasion of a choice rather than the total randomization of pure strategies made before a play by means of a mixed strategy. We will not elaborate further on behavioral strategies because the method developed here is basically self-contained.

Consider the linear program (P):

$$\max \sum_{(i,j)\in C} p_{ij} y_{ij},$$

subject to

$$(j - i)x_{kl} + y_{ij} \le j, \qquad (i, j) \in C, \qquad (k, l) \in D_{ij},$$

$$x_{kl} \le 1,$$

$$x_{kl} \ge 0, \qquad (k, l) \in L. \tag{1}$$

The variables in (P) are x_{kl} for $(k, l) \in L$, and y_{ij} for $(i, j) \in C$. An optimal solution exists for (P) because the nonempty feasible set is closed, and the objective function is bounded from above. Let $\{x_{kl}^*, y_{ij}^*\}$ be an optimal solution, and let z^* be the optimal value of P. We shall show $v = z^*$.

Let P_2 adopt the strategy σ_2, where he selects boxes 1 and 2 with probabilities x_{kl}^* and $1 - x_{kl}^*$, respectively, if the lid labels are (k, l). We claim that, by using σ_2, P_2 can ensure an expected payoff of at least z^*. Suppose P_1 sees the box contents (i, j); this even can arise with probability $p_{ij} > 0$. If P_1 changes (i, j) to some $(k, l) \in D_{ij}$, P_2 will receive, when using σ_2, an expectation $ix_{kl}^* + j(1 - x_{kl}^*)$ for this event. Thus P_2 can ensure an expectation of at least $\min_{(k,l)\in D_{ij}} \{ix_{kl}^* + j(1 - x_{kl}^*)\}$ for this event. We claim that

$$y_{ij}^* = \min_{(k,l)\in D_{ij}} \{ix_{kl}^* + j(1 - x_{kl}^*)\}, \qquad (i, j) \in C. \tag{2}$$

From (1),

$$y_{ij}^* = \min_{(k,l)\in D_{ij}} \{ix_{kl}^* + j(1 - x_{kl}^*)\}, \qquad (i, j) \in C. \tag{3}$$

Suppose the inequality in (3) is strict for some $(i, j) \in C$. Then $y_{ij}^* < ix_{kl}^* + j(1 - x_{kl}^*)$ for each $(k, l) \in D_{ij}$. It is therefore possible to increase this particular y_{ij}^* by a small amount, while keeping the values of all other optimal variables in $\{x_{kl}^*, y_{ij}^*\}$ unchanged, to obtain a feasible solution to (P) with a larger value of the objective function. This contradiction proves (2). Summing over $(i, j) \in C$, P_2 can ensure an expected payoff of at least $\Sigma_{(i,j)\in C}\, p_{ij}y_{ij}^* = z^*$. Hence

$$v \geq z^*. \tag{4}$$

To prove the reverse inequality $v \leq z^*$, we proceed as follows. For $(i, j) \in C$ and $(k, l) \in L$, define

$$\delta_{ij}^{kl} = \begin{cases} 1, & \text{if } (k, l) \in D_{ij}, \\ 0, & \text{otherwise.} \end{cases}$$

The dual of (P) is (D):

$$\min \sum_{(k,l)\in L} s_{kl} + \sum_{(i,j)\in C} \sum_{(k,l)\in D_{ij}} jt_{ij}^{kl}$$

subject to

$$\begin{aligned} & s_{kl} + \sum_{(i,j)\in C} (j - i)\, \delta_{ij}^{kl} t_{ij}^{kl} \geq 0, && (k, l) \in L, \\ & \sum_{(k,l)\in D_{ij}} t_{ij}^{kl} = p_{ij}, && (i, j) \in C, \\ & t_{ij}^{kl} \geq 0, && (i, j) \in C, \quad (k, l) \in D_{ij}, \\ & s_{kl} \geq 0, && (k, l) \in L. \end{aligned} \tag{5}$$

The variables in (D) are s_{kl} for $(k, l) \in L$, and t_{ij}^{kl} for $(i, j) \in C$, $(k, l) \in D_{ij}$. Since (P) has an optimal solution, (D) also has an optimal solution $\{s_{kl}^*, t_{ij}^{kl^*}\}$ with the optimal value of z^*.

Let P_1 adopt the strategy σ_1 where he changes (i, j) to $(k, l) \in D_{ij}$ with probability $t_{ij}^{kl^*}/p_{ij}$ if the box contents are (i, j). We claim P_1 can restrict P_2's expected payoff to at most z^*. Let $E_1(k, l)$ and $E_2(k, l)$ denote the contribution to P_2's expected payoff when he chooses box 1 and box 2, respectively, on seeing (k, l). Then

$$E_1(k, l) = \sum_{(i,j)\in C} p_{ij} i\, \delta_{ij}^{kl} t_{ij}^{kl^*}/p_{ij} = \sum_{(i,j)\in C} i\, \delta_{ij}^{kl} t_{ij}^{kl^*},$$

$$E_2(k, l) = \sum_{(i,j)\in C} p_{ij} j\, \delta_{ij}^{kl} t_{ij}^{kl^*}/p_{ij} = \sum_{(i,j)\in C} j\, \delta_{ij}^{kl} t_{ij}^{kl^*}.$$

Notice, for example, by the definition of $E_1(k, l)$, we have incorporated the probability of (k, l) occurring into the expression. Now P_2 can receive, for each $(k, l) \in L$, at most $\max\{E_1(k, l), E_2(k, l)\}$, so that he can receive an expected payoff of at most $\Sigma_{(k,l)\in L} \max\{E_1(k,l), E_2(k, l)\}$.

We next show this last sum is equal to z^*. For each $(k, l) \in L$ such that $s_{kl}^* > 0$, the inequality in (5) must hold as an equality (note our discussion is confined to these constraints at an optimal solution). If not, we can decrease the value of the objective function by reducing the value of this s_{kl}^* by a small amount. Hence for $s_{kl}^* > 0$, we have $E_1(k, l) - E_2(k, l) = s_{kl}^* > 0$, so that $E_1(k, l) > E_2(k, l)$. From (5), for $s_{kl}^* = 0$, we have $E_2(k, l) \geq E_1(k, l)$. Let $L_+ = \{(k, l) \in L: s_{kl}^* > 0\}$ and $L_0 = \{(k, l) \in L: s_{kl}^* = 0\}$. The optimal value of (D) is

$$\begin{aligned}
z^* &= \sum_{(k,l)\in L} s_{kl}^* + \sum_{(i,j)\in C} \sum_{(k,l)\in D_{ij}} j t_{ij}^{kl^*} \\
&= \sum_{(k,l)\in L} s_{kl}^* + \sum_{(i,j)\in C} \sum_{(k,l)\in L} j\, \delta_{ij}^{kl} t_{ij}^{kl^*} \\
&= \sum_{(k,l)\in L} \left\{ s_{kl}^* + \sum_{(i,j)\in C} j\, \delta_{ij}^{kl} t_{ij}^{kl^*} \right\} \\
&= \sum_{(k,l)\in L} \{s_{kl}^* + E_2(k, l)\} \\
&= \sum_{(k,l)\in L_0} E_2(k, l) + \sum_{(k,l)\in L_+} \{s_{kl}^* + E_2(k, l)\} \\
&= \sum_{(k,l)\in L_0} E_2(k, l) + \sum_{(k,l)\in L_+} E_1(k, l) \\
&= \sum_{(k,l)\in L} \max\{E_1(k, l), E_2(k, l)\}.
\end{aligned}$$

This proves that, by using σ_1, P_1 can restrict P_2's expected payoff to at most z^*. Hence

$$v \leq z^*. \tag{6}$$

From (4) and (6), we have the following theorem.

THEOREM: The value of Γ is $v = z^*$. The strategies σ_1 and σ_2 are optimal for P_1 and P_2, respectively.

We can now address the questions posed earlier. P_1 should distort the information according to σ_1, while P_2 should select a box according to σ_2. Next, consider the case when the distorted information is useless to P_2. Under such a circumstance, he has no choice but to ignore completely the distorted information in his selection of a box. In other words, he does not even have to look at the lid labels. If he always selects box 1 (respectively, box 2), his expected payoff is $\Sigma_{(i,j)\in S}\, ip_{ij}$ (respectively, $\Sigma_{(i,j)\in S}\, jp_{ij}$). By selecting the appropriate box, P_2 can obtain an expected payoff equal to the larger of these two quantities, so that

$$v \geq \max\left\{ \sum_{(i,j)\in S} ip_{ij}, \sum_{(i,j)\in S} jp_{ij} \right\}. \tag{7}$$

If the inequality in (7) holds as an equality, clearly the distorted information is useless to P_2. On the other hand, if this inequality is strict, then the distorted information is useful to P_2. This is because, by using it wisely, he can do better than by selecting the box with the larger mean box contents. Thus we have a simple criterion to determine whether the distorted information is useful to P_2 or not.

3. A PARTICULAR GAME

Let n and r be given integers with $n \geq 2$ and $1 \leq r \leq n - 1$. Let $N = \{(i, j): i, j = 1, \ldots, n\}$. Consider the case of Γ where $S = C = N$, $p_{ij} = 1/n^2$, $D_{ij} = \{(i, h): h \in \{1, \ldots, n\}, |j - h| \leq r\} \cup \{(g, j): g \in \{1, \ldots, n\}$, and $|i - g| \leq r\}$ for $(i, j) \in N$. In other words, the box contents (i, j) are chosen uniformly from N; P_1 is allowed to change only one of the box contents to some integer in $\{1, \ldots, n\}$ not more than r units distance away. We denote this game and its value by Γ_{nr} and v_{nr}, respectively. Note that it is meaningful to define Γ_{nr} for $r \geq n$. But since $\Gamma_{nr} = \Gamma_{n,n-1}$ for $r \geq n$, it suffices to consider $r \leq n - 1$.

Let σ be an optimal strategy of P_1 in Γ_{nr}. If P_1 uses σ in $\Gamma_{n,r+1}$, he can restrict P_2's expected payoff to at most v_{nr}. This implies

$$v_{n,n-1} \leq v_{n,n-2} \leq \cdots \leq v_{n1}. \tag{8}$$

A lower bound on v_{nr} can be easily obtained from (7); that is,

$$v_{nr} \geq (n + 1)/2, \qquad 1 \leq r \leq n - 1. \tag{9}$$

Our object now is to solve Γ_{nr} for various values of n and r, subject to $n \geq 2$ and $1 \leq r \leq n - 1$. Given fixed n and r, we need, in general, to solve either (P) or (D) to obtain a solution to Γ_{nr}. We will give an explicit solution of Γ_{nr} for n arbitrary, and the four cases $r = 1, 2, n - 2, n - 1$. Notice these values of r are near the endpoints 1 and $n - 1$, these cases being the simplest ones to solve. Computational experience suggests a solution of Γ_{nr} is quite involved for r near the midpoint of 1 and $n - 1$. As before, we denote the box contents by (i, j), and the lid labels by (k, l). To simplify the description of the players' strategies in the proofs of Propositions 1–4 below, we introduce the following convention and notation. When describing a strategy (sometimes called a response) of P_1, we have to indicate how he changes each $(i, j) \in N$. For brevity, we will only indicate those (i, j) that are *actually* changed. The remaining (i, j) not explicitly mentioned are assumed to be left unchanged by P_1. We say P_1 *collapses* B (a subset of N) to mean P_1 changes each $(i, j) \in B$ with $i \neq j$ to (i, i) or (j, j), depending on whether $i < j$ or $i > j$. We denote by σ_b the strategy of P_2 where he selects box 1 if $k \geq l$, and box 2 if $k < l$. Basically, σ_b tells P_2 to select the box with the bigger lid label. Let $N_r = \{(i, j) \in N: |i - j| \leq r\}$.

PROPOSITION 1: $v_{n1} = a_n = (4n^3 + 3n^2 - 7n + 6)/6n^2$, $n \geq 2$.

PROOF: Let P_2 adopt σ_b. It is easy to check that a best response from P_1 is to change $(i - 1, i)$ to (i, i), $i = 2, \ldots, n$. Simple calculations show P_2 will receive an expected payoff of a_n against this best response. Since P_2 can ensure an expected payoff of at least a_n, we have

$$v_{n1} \geq a_n. \tag{10}$$

Let P_1 adopt the strategy where he collapses N_1. It is again easy to check P_1 can restrict P_2's expected payoff to at most a_n, so that

$$v_{n1} \leq a_n. \tag{11}$$

The result then follows from (10) and (11).

PROPOSITION 2: $v_{n,n-1} = (n + 1)/2$, $n \geq 2$.

PROOF: Let P_1 adopt the strategy where he collapses N. Using this strategy, P_1 can restrict P_2's expected payoff to $(n + 1)/2$, so that $v_{n,n-1} \leq (n + 1)/2$. The result then follows from (9). Note that to always select box 1 is an optimal strategy of P_2.

PROPOSITION 3: Let

$$b_n = \begin{cases} 20/9, & \text{if } n = 3, \\ 21/8, & \text{if } n = 4, \\ (n + 1)/2, & \text{if } n \geq 5. \end{cases}$$

Then $v_{n,n-2} = b_n$, $n \geq 3$.

PROOF: The case $n = 3$ follows from Proposition 1.

Next, consider the case $n = 4$. Let P_2 adopt the strategy where he selects box 1 if $(k, l) = (2, 3)$; otherwise, he selects the box according to σ_b. A best response from P_1 is to change (3, 2) to (1, 2), (1, 2) and (3, 4) to (3, 2), (1, 3), and (2, 4) to (2, 3). Simple calculations show P_2 can ensure an expected payoff of at least $\frac{21}{8}$, so that $v_{42} \geq \frac{21}{8}$. Let P_1 adopt the strategy where he changes (1, 4) and (3, 2) to (1, 2), (2, 3) and (4, 1) to (2, 1), and collapses {(1, 2), (1, 3), (2, 1), (2, 4), (3, 1), (3, 4), (4, 2), (4, 3)}. It is easy to verify $v_{42} \leq \frac{21}{8}$.

Consider now the case $n \geq 5$. Let P_1 adopt the strategy where he changes

$(1, n)$ to $(1, 2)$ with probability $\frac{1}{2}$,
$(n - 1, 2)$ to $(1, 2)$ with probability $\frac{1}{2}$,
$(3, 2)$ to $(1, 2)$,
$(1, n)$ to $(n - 1, n)$ with probability $\frac{1}{2}$,
$(n - 1, 2)$ to $(n - 1, n)$ with probability $\frac{1}{2}$,
$(n - 1, n - 2)$ to $(n - 1, n)$,
$(n, 1)$ to $(2, 1)$ with probability $\frac{1}{2}$,
$(2, n - 1)$ to $(2, 1)$ with probability $\frac{1}{2}$,
$(2, 3)$ to $(2, 1)$,
$(n, 1)$ to $(n, n - 1)$ with probability $\frac{1}{2}$,
$(2, n - 1)$ to $(n, n - 1)$ with probability $\frac{1}{2}$,
$(n - 2, n - 1)$ to $(n, n - 1)$, and
collapses the remaining pairs in N_{n-2}.

For each fixed $(k, l) \in \{(1, 2), (2, 1), (n - 1, n), (n, n - 1), (1, 1), \ldots, (n, n)\}$, P_1 will receive the same expectation, independent of the box he chooses. This implies P_1 can restrict P_2's expected payoff to $(n + 1)/2$, so that $v_{n,n-2} \leq (n + 1)/2$. The result then follows from (9).

REMARK: When $n = 4$ in Proposition 3, σ_b is not an optimal strategy of P_2. If P_2 adopts σ_b, a best response from P_1 is to change (1, 3) to (1, 1), (2, 3) to (2, 1), (2, 4) to (2, 2), (3, 2) to (1, 2), (1, 2) and (3, 4) to (3, 2), (2, 1), and (4, 3) to (2, 3). Thus P_2 can only ensure an expected payoff of $\frac{5}{2}$, which is less than $\frac{21}{8}$.

PROPOSITION 4: Let

$$c_n = \begin{cases} 2, & \text{if } n = 3, \\ 21/8, & \text{if } n = 4, \\ (4n^3 + 3n^2 - 25n + 36)/6n^2, & \text{if } n \geq 5. \end{cases}$$

Then $v_{n,2} = c_n$, $n \geq 3$.

PROOF: The case $n = 3$ follows from Proposition 2, and the case $n = 4$ from Proposition 3.

Consider now the case $n \geq 5$. Let P_2 adopt σ_b. It is easy to check a best response from P_1 is to change $(1, 2)$ to $(3, 2)$, $(n, n - 1)$ to $(n - 2, n - 1)$, $(i, i + 2)$ to (i, i), $(i + 1, i)$ to $(i + 1, i + 2)$, $(i + 1, i + 2)$ to $(i + 1, i)$, $i = 1, \ldots, n - 2$. Simple calculations show P_2 will receive an expected payoff of c_n against this best response. Since P_2 can ensure an expected payoff of at least c_n, we have $v_{n2} \geq c_n$, $n \geq 5$.

It is routine to verify the following strategies of P_1 can restrict P_2's strategy to at most c_n, so that $v_{n2} \le c_n$, $n \ge 5$. In the strategies below, when we say P_1 changes (i_0, j_0) to (k_0, l_0), it is implicitly assumed he also changes (j_0, i_0) to (l_0, k_0).

When $n = 5$, P_1 changes (2, 1), (2, 5), (4, 3) to (2, 3), changes (1, 4), (3, 2), (5, 4) to (3, 4), and collapses the remaining pairs in N_2.

When $n = 6$, P_1 changes (2, 1) with certainty, (2, 5) with probability $\frac{1}{2}$, (4, 3) with probability $\frac{1}{2}$ to (2, 3), changes (3, 2), (3, 6), (5, 4) to (3, 4), changes (6, 5) with certainty, (2, 5) with probability $\frac{1}{2}$, (4, 3) with probability $\frac{1}{2}$ to (4, 5), and collapses the remaining pairs in N_2.

When $n \ge 7$, P_1 changes

(2, 1) to (2, 3), $(n, n-1)$ to $(n-2, n-1)$,
$(i, i+3)$ and $(i+2, i+1)$ to $(i, i+1)$, $i = 1, \ldots, \lfloor n/2 \rfloor - 1$,
$(i, i+3)$ and $(i+2, i+1)$ to $(i+2, i+3)$, $i = \lfloor n/2 \rfloor, \ldots, n-3$, and
collapses the remaining pairs in N_2. This completes the proof.

Let n be fixed. From (8), as r increases from 1 to $n - 1$, v_{nr} decreases (not necessarily strictly) from a_n to $(n + 1)/2$. Let m be the smallest integer such that $v_{nm} = (n + 1)/2$; m exists by Proposition 2. It follows $v_{nr} = (n + 1)/2$ for $r > m$. The implication here is that, if we solve Γ_{nr} in terms of increasing r, we can stop as soon as we find the first r with $v_{nr} = (n + 1)/2$. In Γ_{nr}, the distorted information is useful or not to P_2, depending whether $r < m$ or $r \ge m$. Given n, the value of m can be obtained by solving Γ_{nr} for a few values of r chosen iteratively, say, by a bisection method. For example, when $n = 5$, we have $m = 3$.

4. CONCLUSIONS

The primary purpose of this article is to investigate how a person P_2 can make a good decision based on some distorted information. In particular, we would like to identify the condition under which this information is useful to him. We assume the worst-case scenario—the information has been deliberately distorted by a clever adversary P_1 to penalize him as much as possible. This assumption may be questionable in some situations. However, if P_2 finds the information distorted from such an adversary useful, then of course he can be certain of the usefulness of such information distorted by nature or lesser adversaries. We formulate a simple game with distorted information in terms of two boxes in Section 2, and show how to solve it using linear programming. Note that the method there can be easily extended to solve the problem with any finite number of boxes, each box containing one number. We have not analyzed this more general problem due mainly to the complexity in notation. In Section 3, we discuss in greater detail a particular game which is nontrivial to solve explicitly.

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